70 KSME Journal, Vol. 7, No. 1, pp. 70~75, 1993

Curve Veering in Outer-Clamped Annular Plates Subject to Nonuniform In-Plane Force

Kyung-Hwa Rim* and Chong-Won Lee**

(Received August 3, 1992)

The existence of curve veering phenomenon is investigated in outer-clamped annular plates subject to in-plane force. The perturbation and Galerkin's methods are employed to verify the existence qualitatively and quantitatively, respectively. From qualitative and quantitative analyses, it is found that the curve veering phenomenon may take place in outer-clamped annular plates under nonuniform in-plane force.

Key Words: Curve Veering, In-Plane Force, Annular Plate, Perturbation Method, Galerkin's Method

1. Introduction

Curve veering refers to the phenomenon observed in plots of two adjacent natural frequencies versus a parameter of a dynamic system. At a point where the curves of natural frequencies appear that they should cross, they may instead approach one another quite closely and then "veer" away. In the veering region, the modes undergo abrupt changes in mode shapes, exchange mode shapes and subsequently follow path previously taken by the other curve. The check about the existence of cure veering is necessary in explaining unexpected changes in calculated natural frequencies, measuring mode shapes and considering the mode shapes in design.

Researches about curve vering were actually initiated by Leissa(1974). Having reviewed a number of papers which exhibited the curve veering aberration associated with plate vibraions, he raised the question of whether curve veering occurs due to the error from mathematical models or approximate methods. But Kuttler and Sigillito(1981) asserted, by using the upper and lower bounds of natural frequencies, that curve veering can be an actual phenomenon of the mathematical model. Schajer(1984) showed the curve veering phenomenon in the discrete model of a rotating circular string. Perkins and Mote(1986) established the criteria to distinguish curve veering from crossing in both continuous and discrete models. The existence of curve veeirng, in continuous models, is also illustrated by utilizing the exact solution of an elementary eigenvalue problem. Recently Jei and Lee(1990) verified the existence of curve veering in rotor-bearing systems by using the criteria presented by Perkins and Mote(1986). In addition, they showed, by using the exact solution of continuous rotor models, that the curve veering can take places as the bearing stiffness and the rotational speed change.

In this paper, the existence of curve veering in continuous model of annular plates subject to in-plane force is investigated by using the criteria presented by Perkins and Mote(1986). Curve

* Samsung Advanced Institute of Technology P.O. Box 111, Suwon 440-600, Korea
** Center for Noise and Vibration Control(NOVIC) Department of Mechanical Engineering Korea Advanced Institute of Science and Technology Science Town, Taejon 305-701, Korea

veering phenomenon is also examined as the magnitude of nonuniform in-plane force varies, through numerical analysis using Galerkin's method.

2. Equation of Motion

Figure 1 shows the schematic diagram of an annular plate which is free at the inner radius $R=a$ and clamped at the outer radius $R=b$. And the annular plate is subject to arbitrary inplane force, $P(\theta)$, along the outer edge, which is a function of spatial variable θ but assumed to be time-independent.

Assuming small transverse deflections and neglecting the effect of damping, rotary inertia and shear deformation, the equation of motion of the annular plate, which is isotropic, homogeneous and of uniform thickness, may be written in a nondimensionalized form as

$$\bar{\nabla}^4 w-\bar{N}_r\frac{\partial^2 w}{\partial r^2}-\bar{N}_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$-2\bar{N}_{r\theta}\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)-\lambda w=0, \qquad (1)$$

where

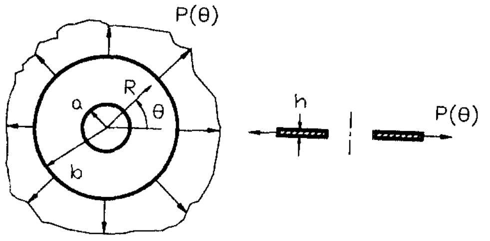
$$w=\frac{W}{b},\ r=\frac{R}{b},\ \bar{N}_\gamma=\frac{N_\gamma b^2}{D}(\gamma=r,\ \theta,\ r\theta),$$
$$D=\frac{Eh^3}{12(1-\nu^2)},$$
$$\bar{\nabla}^4=(\bar{\nabla}^2)^2=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2,$$
$$\lambda=\frac{\omega^2 b^4\rho h}{D}.$$

Here $W(R,\ \theta)$ is the transverse deflection, h the thickness, E the elastic modulus, D the flexural rigidity, ν the Poisson's ratio ρ the density and ω the circular frequency. And $N_r(R,\theta)$, $N_\theta(R,\theta)$ and $N_{r\theta}(R,\theta)$ are the stress resultants in thickness. The associated boundary conditions become at the clamped outer radius, $r=1$,

$$w=0,\qquad \frac{\partial w}{\partial r}=0, \qquad (2a)$$

and at the free inner radius, $r=\alpha=a/b$

$$M_r=0,\quad V_r=0, \qquad (2b)$$



a plane figure   a cross sectional view

Fig. 1 Schematic figuration of an annular plate under in-plane force

where M_r and V_r denote the bending moment and vertical force per unit length, respectively, associated with the cross section whose normal is r.

The arbitrary in-plane force, which is a periodic function with a period of 2π, can be expanded in a nondimensional Fourier series as

$$\bar{P}(\theta)=A_0+\sum_{q=1}^{\infty}(A_q\cos q\theta+B_q\sin q\theta), \qquad (3)$$

where

$$\bar{P}(\theta)=\frac{P(\theta)b^2}{D}$$

and

$$A_o=\frac{1}{2\pi}\int_0^{2\pi}\bar{P}(\theta)d\theta,$$
$$A_q=\frac{1}{\pi}\int_0^{2\pi}\bar{P}(\theta)\cos q\theta\ d\theta,$$
$$B_q=\frac{1}{\pi}\int_0^{2\pi}\bar{P}(\theta)\sin q\theta\ d\theta,$$
$$q=1,\ 2\cdots$$

Imposing the static equilibrium conditions given by

$$\int_0^{2\pi}\bar{P}(\theta)\cos\theta\ d\theta=0,$$
$$\int_0^{2\pi}\bar{P}(\theta)\sin\theta\ d\theta=0, \qquad (4a)$$

one obtains

$$A_1=B_1=0. \qquad (4b)$$

The stress field of an annular plate under arbitrary in-plane force can be obtained by using Airy stress resultant function, $\Phi(r,\ \theta)$, in plane stress conditions. The stress resultants are given in terms of the Airy stress resultant function, for the case of no body force, by

$$\bar{N}_r=\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2},$$
$$\bar{N}_\theta=\frac{\partial^2 \Phi}{\partial r^2},$$
$$\bar{N}_{r\theta}=-\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial \Phi}{\partial \theta}\right)\right]. \tag{5}$$

The solutions of the Airy stress resultant function can be obtained by considering single valued stresses and displacements, and substituting the boundary conditions into general solution of the equation, $\nabla^4\Phi=0$. The boundary conditions for outer-clamped annular plate subject to in-plane force are given as, at $r=1$,

$$\bar{N}_r=\bar{P}(\theta),\quad \bar{N}_{r\theta}=0, \tag{6a}$$

and at $r=a$

$$\bar{N}_r=0,\quad \bar{N}_{r\theta}=0. \tag{6b}$$

Substituting the Airy stress resultant function into Eq. (5), the stress resultants are obtained as

$$\bar{N}_r(r,\theta)=A_0H_r(r,0)+\sum_{q=2}^{\infty}H_r(r,q)$$
$$[A_q\cos q\theta+B_q\sin\theta],$$
$$\bar{N}_\theta(r,\theta)=A_0H_\theta(r,0)+\sum_{q=2}^{\infty}H_\theta(r,q)$$
$$[A_q\cos q\theta+B_q\sin\theta],$$
$$\bar{N}_{r\theta}(r,\theta)=\sum_{q=2}^{\infty}H_{r\theta}(r,q)$$
$$[A_q\sin q\theta-B_q\cos q\theta], \tag{7}$$

where

$$H_r(r,0)=\frac{1}{1-a^2}\left(1-(\frac{a}{r})^2\right),$$
$$H_\theta(r,0)=\frac{1}{1-a^2}\left(1+(\frac{a}{r})^2\right),$$

and for $q\neq 0$

$$H_r(r,q)=\frac{1}{G}[c_1qr^{q-2}-c_2(q-2)r^q$$
$$-c_3qr^{-q-2}+c_4(q+2)r^{-q}],$$
$$H_\theta(r,q)=\frac{1}{G}[-c_1qr^{q-2}+c_2(q+2)r^q$$
$$+c_3qr^{-q-2}-c_4(q-2)r^{-q}],$$
$$H_{r\theta}(r,q)=\frac{q}{G}[-c_1r^{q-2}+c_2r^q$$
$$-c_3r^{-q-2}+c_4r^{-q}],$$
$$G(q,a)=2[(a^{2q}-1)^2-q^2a^{2q}(a-a^{-1})^2],$$
$$c_1(q,a)=1+a^{2q}(-qa^2+q-1),$$
$$c_2(q,a)=1+a^{2q}(qa^{-2}-q-1),$$
$$c_3(q,a)=a^{2q}(a^{2q}+qa^2-q-1),$$
$$c_4(q,a)=a^{2q}(a^{2q}-qa^{-2}+q-1).$$

3. Qualitative Verification by Perturbation Method

A perturbation method is used to verify the existence of curve veering qualitatively, in outer-clamped annular plates subject to in-plane force. The perturbation of in-plane force, $\Delta P(\theta)$, in eigenvalue problem is considered to investigate the change of modal parameters according to the variation of in-plane force. Eigenvalue problem of an annular plate under in-plane force can be expressed, by using the self-adjoint and positive definite operators L, M and B_γ for in-plane force $P(\theta)$, as

$$L[\phi_i]=\lambda_iM[\phi_i],\ 0<\theta\leq 2\pi,\ a\leq r\leq 1,$$
$$B_\gamma[\phi_i]=0,\ \text{at } r=1$$
$$\text{and } r=a;\ \gamma=1,2,3,4, \tag{8}$$

where

$$\lambda_i=\frac{\omega_i^2b^4\rho h}{D},$$
$$L[\phi]=\bar{\nabla}^4\phi-\bar{N}_r\frac{\partial^2\phi}{\partial r^2}-\bar{N}_\theta\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right.$$
$$\left.+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)-2\bar{N}_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right),$$
$$M[\phi]=\phi.$$

Here ϕ_i and ω_i are the i-th mode shape (eigenfunction) and natural frequency, respectively. And the boundary operators B_γ are derived from Eqs. (2a) and (2b).

Note that the perturbation $\Delta P(\theta)$ in in-plane force results in the perturbation ΔL only in the operator L. To evaluate the criteria of the curve veering in the eigenvalue problem, assume that there exist two nearly equal eigenvalues, $\dot{\lambda}_i$ and $\dot{\lambda}_j$, in the unperturbed problem, satisfying

$$|\dot{\lambda}_i-\dot{\lambda}_j|\ll\min_{s,t\neq i,j}|\dot{\lambda}_s-\dot{\lambda}_t|. \tag{9}$$

By using perturbation method (Perkins and Mote, 1986), the considered eigenvalues can be perturbed, to the second order, as

$$\lambda_i=\dot{\lambda}_i+d_{ii}+\frac{d_{ij}^2}{\dot{\lambda}_i-\dot{\lambda}_j},$$
$$\lambda_i=\dot{\lambda}_i+d_{jj}-\frac{d_{ij}^2}{\dot{\lambda}_i-\dot{\lambda}_j}, \tag{10}$$

where

$$\begin{aligned} d_{ij}=d_{ji}&=<\Delta L(\dot{\phi}_j),\ \dot{\phi}_i> \\ &=\int_0^{2\Pi}\int_a^1\Big[\Delta\bar{N}_r\frac{\partial\dot{\phi}_i}{\partial r}\frac{\partial\dot{\phi}_j}{\partial r}r \\ &+\Delta\bar{N}_\theta\frac{\partial\dot{\phi}_i}{\partial\theta}\frac{\partial\dot{\phi}_j}{\partial\theta}\frac{1}{r} \\ &+\Delta\bar{N}_{r\theta}\Big(\frac{\partial\dot{\phi}_i}{\partial r}\frac{\partial\dot{\phi}_j}{\partial\theta}+\frac{\partial\dot{\phi}_i}{\partial\theta}\frac{\partial\dot{\phi}_j}{\partial r}\Big)\Big]drd\theta,\end{aligned}$$

and $\Delta\bar{N}_\gamma(\gamma=r,\ \theta,\ r\theta)$ are the perturbation of stress resultants due to $\Delta P(\theta)$. With the expansion of d_{ij} in Taylor series about $P(\theta)=P_o(\theta)$ in Eq. (10), the perturbed eigenvalues are obtined as

$$\lambda_i=\dot{\lambda}_i+(D^1d_{ii})\,\varepsilon+\frac{1}{2}\frac{1}{\dot{\lambda}_i-\dot{\lambda}_j}(D^2d_{ij}^2)\,\varepsilon^2,$$
$$\lambda_j=\dot{\lambda}_j+(D^1d_{jj})\,\varepsilon-\frac{1}{2}\frac{1}{\dot{\lambda}_i-\dot{\lambda}_j}(D^2d_{ij}^2)\,\varepsilon^2, \tag{11}$$

where $D^1=\frac{\partial}{\partial P}|_{P_O}$, $D^2=\frac{\partial^2}{\partial P^2}|_{P_o}$ and ε is the norm of perturbation $\Delta P(\theta)$. Note that $D^2d_{ij}^2$ measures the coupling of the unpertubed modes. Without coupling, $D^2d_{ij}^2=0$, λ_i and λ_j are locally independent and free to cross, but with coupling, $D^2d_{ij}^2\neq0$, loci concavities depend strongly on their separation $|\dot{\lambda}_i-\dot{\lambda}_j|$. Thus the value of $D^2d_{ij}^2$ indicates crossing or veering in the vicinity of two nearly equal natural frequencies. Since the terms in ε^2 order have opposite sign in case of $D^2d_{ij}^2\neq0$ in Eq. (11), the loci of λ_i and λ_j veer towards each other.

In order to verify the existence of curve veering phenomenon in annular plates under in-plane force, consider the case of $\Delta P(\theta)=\Delta A_q\cos q\theta$, that is, the perturbation in a single Fourier series component of in-plane force. From the results of qualitiative analysis (Rim and Lee, 1991), unperturbed modes with nodal diameters consist of a pair of (m,n)S and (m,n)A taking symmetric and anti-symmetric mode shapes, respectively, with respect to one of the major principal axes associated with initial in-plane force $P_0(\theta)=A_{q0}\cos q\theta$, *i.e.* $\theta=0$ here. And unperturbed modes without nodal diameters, $(0,n)$, are classified as (m,n)S modes due to symmetric property of associated mode shapes. Thus the unperturbed mode shapes $\dot{\phi}_i$'s can be expressed as

$$\dot{\phi}_i(r,\ \theta)=F_i(r)\,G_i(\theta), \tag{12}$$

where $G_i(\theta)$ is the function symmetric or anti-symmetric with respect to $\theta=0$. When $\Delta P(\theta)=\Delta A_q\cos q\theta$, $D^2d_{ij}^2$ is given as

$$\begin{aligned}\frac{1}{2}[D^2d_{ij}^2]=\Big[&\int_a^1 H_r(r,\ q)\frac{dF_i(r)}{dr}\frac{dF_j(r)}{dr}rdr \\ &\int_0^{2\pi}G_i(\theta)\,G_j(\theta)\cos q\theta d\theta \\ &+\int_a^1 H_\theta(r,\ q)\,F_i(r)\,F_j(r)\frac{1}{r}dr \\ &\int_0^{2\pi}\frac{dG_i(\theta)}{d\theta}\frac{dG_j(\theta)}{d\theta}\cos q\theta d\theta \\ &+\int_a^1 H_{r\theta}(r,\ q)\,F_i(r)\frac{dF_j(r)}{dr}dr \\ &\int_0^{2\pi}\frac{dG_i(\theta)}{d\theta}G_j(\theta)\sin q\theta d\theta \\ &+\int_a^1 H_{r\theta}(r,\ q)\,F_j(r)\frac{dF_i(r)}{dr}dr \\ &\int_0^{2\pi}\frac{dG_j(\theta)}{d\theta}G_i(\theta)\sin q\theta d\theta\Big]^2.\end{aligned} \tag{13}$$

where $H_\gamma(r,\ q)$ $(\gamma=r,\ \theta,\ r\theta)$ associated with stress fields is given in Eq. (7).

Now consider two cases:perturbations in uniform $(q=0)$ and nonuniform $(q\neq0)$ in-plane forces. When $q=0$, the vanlue of $D^2d_{ij}^2$ vaishes in all modes. Thus curve veering does not take place in annular plates under uniform in-plane force. When $q\neq0$, the value of $D^2d_{ij}^2$ depends on the symmetrical property, with respect to $\theta=0$, of mode shapes, $\dot{\phi}_i$ and $\dot{\phi}_j$. When $\dot{\phi}_i$ and $\dot{\phi}_j$ are both (m,n)S or both (m,n)A mode shapes, $D^2d_{ij}^2$ may not be zero. Then the curve veering can occur. In the other cases, $D^2d_{ij}^2$ always becomes zero. Thus curve veering does not take place between (m,n)S and (m,n)A modes.

4. Quantitative Verification by Galerkin's Method

Since the exact solution of Eq. (1) does not exist, Galerkin's method is employed. In order to apply Galerkin's method to eigenvalue problem, consider a complete set of comparison functions which are the mode shapes of an annular plate without in-plane force. Then the mode shapes of an annular plate under in-plane force can be approximated, by using the pairs of comparison function, $u^c_{mn}(r, \theta)$ and $u^s_{mn}(r, \theta)$, as

$$\phi(r, \theta)=\sum_{m=0}^{m'}\sum_{n=0}^{n'}[a_{mn}u^c_{mn}(r, \theta)+b_{mn}u^s_{mn}(r, \theta)], \tag{14}$$

where

$$u^c_{mn}(r, \theta)=R_{mn}(r)\cos m\theta,\ u^s_{mn}(r, \theta)=R_{mn}(r)\sin m\theta.$$

Here a_{mn}'s and b_{mn}'s are the coefficients of comparison functions, and $m(m')$ and $n(n')$ are the number (maximum number) of nodal diameters and nodal circles, respectively, of (m,n) mode. And $R_{mn}(r)$ consists of Bessel functions. By using the assumed mode shapes in Eq. (14), the Galerkin's integral can be expressed as

$$\int_0^{2\pi}\int_a^1\left\{\begin{matrix}u^c_{gh}(r, \theta)\\ u^s_{gh}(r, \theta)\end{matrix}\right\}[L(\phi)-\lambda M(\phi)]rdrd\theta=\left\{\begin{matrix}0\\0\end{matrix}\right\},$$

$$g=1,2, \cdots m',\ h=1,2, \cdots n', \tag{15}$$

yielding the matrix equation as

$$\boldsymbol{K}\boldsymbol{x}_i=\lambda_i\boldsymbol{M}\boldsymbol{x}_i \tag{16}$$

where the size of mass and stiffness matrices, $\boldsymbol{K}$ and $\boldsymbol{M}$, associated with operators, L and M, respectively, becomes $2(m'+1)(n'+1)\times 2(m'+1)(n'+1)$ and

$$\lambda_i=\frac{\omega_i^2 b^4 \rho h}{D},$$

$$\boldsymbol{x}_i^T=\{a^{(i)}_{00}, a^{(i)}_{01}, \cdots, a^{(i)}_{mn}, b^{(i)}_{00}, b^{(i)}_{01}, \cdots, b^{(i)}_{mn}\}.$$

Here ω_i and $\boldsymbol{x}_i$ are the natural frequency and the coefficient vector of comparison function associated with the i-th mode, respectively.

Now natural frequencies of outer-clamped annular plates under nonuniform in-plane force $A_2\cos 2\theta$ is investigated so as to show the curve veering phenomenon. The results are described in nondimensional form and the following numerical values have been used: Poisson's ratio, $\nu=0.3$; radius ratio, $a=1/3$. The integration process in radial directiion uses the ten-point Gauss Legendre integration method with five intervals.

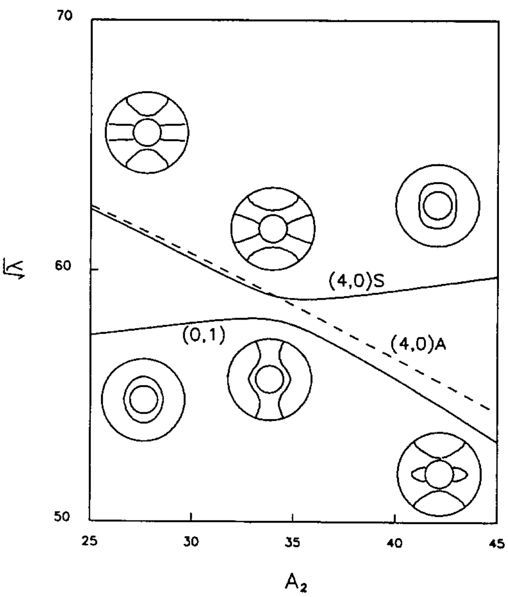


Fig. 2 Natural frequencies and mode shapes in case of $P(\theta)=A_2\cos 2\theta$
:———, veering modes ((0,1),(4,0)S)
;– – –, other mode ((4,0)A)

Figure 2 illustrates the curve veering phenomenon in the variation of natural frequencies and mode shapes as nonuniform in-plane force increases. These are curve veering between (0, 1) and (4, 0)S, and, crossing between (4, 0)S and (4, 0)A. The mode shapes represented by the nodal lines change abruptly in the transient zone of curve veering. The symmetrical properties of modes which induce curve veering in Fig. 2 accord with those presented in perturbation method.

5. Conclusions

The existence of curve veering phenomenon is qualitatively and quantitatively investigated in outer-clamped annular plates under in-plane force, by using the perturbation and Galerkin's methods, respectively.

By use of criteria established by Perkins and Mote(1986), the existence of curve veerings is verified in the eigenvalue problem associated with the continuous model of an annular plate under nonuniform in-plane force. And the numerical simulation using Galerkin's method confirms that the curve veering phenomenon can take place as the magnitude of nonuniform in-plane force varies.

References

Jei, Y.G. and Lee, C.W., 1990, "Curve Veering in the Eigenvalue Problem of Rotor-Bearing Systems," KSME J., Vol. 4, No. 2, pp. 128∼135.

Kuttler, J.R. and Sigillito, V.G., 1981, "On Curve Veering," J. of Sound and Vibration, Vol. 75, No.4, pp. 585∼588.

Leissa, A.W., 1974, "On a Curve Veering Aberration," J. of Applied Mathematics and Physics(ZAMP), Vol. 25, pp. 99∼111.

Perkins, N.C. and Mote, Jr., C.D., 1986, "Comments on Curve Veering in Eigenvalue Problems," J. of Sound and Vibration, Vol. 106, No. 3, pp. 451∼463.

Rim, K.H. and Lee, C.W., 1991, "Free Vibration of Outer-Clamped Annular Plates Subject to Arbitrary In-Plane Force," KSME Spring Annual Meeting, pp. 131∼134.

Schajer G.S., 1984, "The Vibration of a Rotating Circular String Subject to a Fixed Elastic Restraint," J. of Sound and Vibration, Vol. 92, No. 1, pp. 11∼19.